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DAMAGE SIMULATION OF HETEROGENEOUS SOLIDS BY NONLOCAL FORMULATIONS ON ORTHOGONAL GRIDS

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Abstract. *The present paper is part of a comprehensive approach of grid-based modeling. This approach includes geometrical modeling by pixel or voxel models, advanced multiphase B-spline finite elements of variable order and fast iterative solver methods based on the multigrid method. So far, we have only presented these grid-based methods in connection with linear elastic analysis of heterogeneous materials. Damage simulation demands further considerations. The direct stress solution of standard bilinear finite elements is severely defective, especially along material interfaces. Besides achieving objective constitutive modelling, various nonlocal formulations are applied to improve the stress solution. Such a corrective data processing can either refer to input data in terms of Young's modulus or to the attained finite element stress solution, as well as to a combination of both. A damage-controlled sequentially linear analysis is applied in connection with an isotropic damage law. Essentially by a high resolution of the heterogeneous solid, local isotropic damage on the material subscale allows to simulate complex damage topologies such as cracks. Therefore anisotropic degradation of a material sample can be simulated. Based on an effectively secantial global stiffness the analysis is numerically stable. The iteration step size is controlled for an adequate simulation of the damage path. This requires many steps, but in the iterative solution process each new step starts with the solution of the prior step. Therefore this method is quite effective. The present paper provides an introduction of the proposed concept for a stable simulation of damage in heterogeneous solids.*

1 INTRODUCTION

Uniform, orthogonal finite element grids are advantageous for geometrical modelling [4] and application of the multigrid method with respect to both, memory demand and computation times [5]. But as a disadvantage of such finite element grids the strains and stresses at an unaligned interphase of material constituents will be severely defective which is a relevant problem with respect to damage simulation. There are meshing techniques as well as advanced finite elements for adequate modeling of material interfaces. However, the geometrical description of phase topologies with variable shape is very complex, especially with regard to three-dimensional models. The following approach is based on a projection of a heterogeneous material on a grid of standard displacement-based four-node finite elements. Strains/stresses are corrected or improved by applying nonlocal formulations. It is further observed that improved strains/stresses can also be achieved by prior nonlocal formulation of the material properties. The principle feasibility of accurate damage simulation on orthogonal grids is the main focus of this article. A simple isotropic material law for damage in tension without plasticity is applied. However, based on the high resolution of the model, complex geometries of damage can develop on the material level, which is also labeled as the mesoscale. Effectively this can correspond to a complex, anisotropic material behaviour on the macroscale. It is the basic idea and the challenge of this research to replace a complex constitutive law on the macroscale by a simpler and more essential constitutive law on the mesoscale. In fact, therefore a close relationship to the material configuration on the mesoscale would be established. Hereafter, heterogeneous solids with various random or deterministic parameters of material phases can be tested numerically. For this objective it is sufficient to analyze only square specimen of the heterogeneous solid such that the finite element formulation can be restricted to orthogonal meshes.

2 GRID-BASED DATA STRUCTURE

In arbitrary finite element meshes a node table with nodal coordinates and an element table which stores element type and corresponding interconnected nodes is required. This results in numerous different element geometries. The general finite element problem further demands to build and store a global stiffness matrix. Based on a predefined mesh topology of $n_{ex} \times n_{ey}$ finite elements and $n_{nx} \times n_{ny}$ nodes, the computational effort and memory demand of the grid-based finite element problem is considerably reduced. The multigrid method allows for an efficient iterative solution process and with a uniform geometry of elements the computation of a global stiffness matrix is superseded. With respect to the nonlocal formulation, also distances and directions between nodes or elements can effectively be determined. The grid-based formulation finds its direct counterpart in the data structure of the computational implementation. Any value which is connected to elements or nodes is uniquely identified by its position on the grid[1]. Such values refer to displacements, type and values of boundary condition, values of postprocessing and so on. For the linear problem a corresponding listing in connection with the memory demand is documented in [5]. For the present approach some essential data layers are introduced. Figure 2.1 (a) shows the material type. A usual pixel image can be read into the finite element program and each different color will be interpreted as an individual material type. One constitutive law and one Poisson's ratio can be assigned to each material type. The material type

[1] In the computational implementation the grid is handled as a vector. Position (i, j) on the grid is $n = i + jn_x$ in the vector with $i = 0 \dots n_x - 1$ and $j = 0 \dots n_y - 1$.

identifies the different materials in the heterogeneous solid and generally its number is countable. In Fig. 2.1 (a) the three material types define inclusions, matrix material and interfacial transition zones. Figure 2.1 (b) shows the defined Young's modulus which is principally decoupled from the material type. One can assign a specific Young's modulus to all elements of a certain material type, but it also possible to modify the Young's modulus in any way, e.g. for modeling some randomness. In Figure 2.1 (c) each color indicates an area of neighbouring finite elements of the same material type. The data of Fig. 2.1 (c) is generated automatically from Fig. 2.1 (a). It is applied to the nonlocal approach where it is useful to average stress values only within a considered inclusion and neglect values of the matrix material or neighboring inclusions.

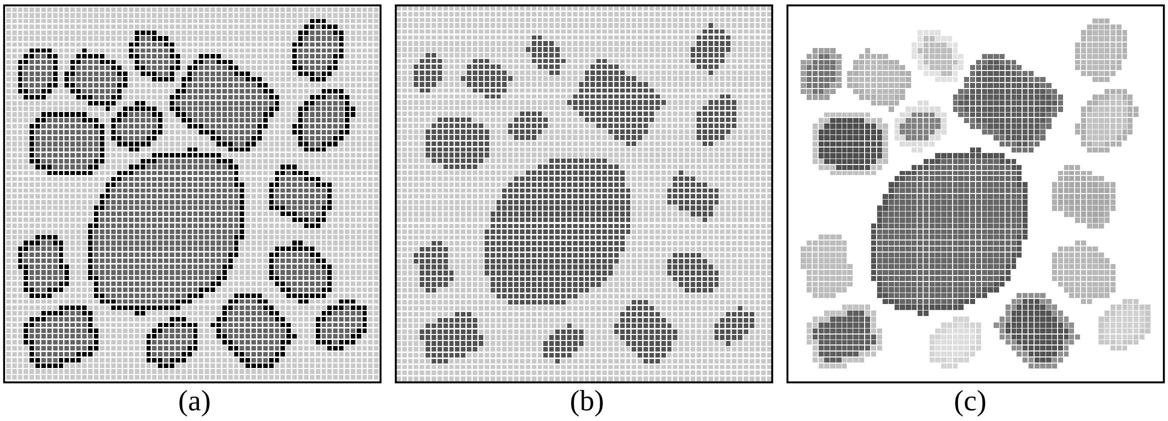

Fig. 2.1: Small heterogeneous material sample of 64×64 finite elements in terms of (a) material type, (b) Young's modulus and (c) individual phases.

3 ISOTROPIC DAMAGE MODEL FOR TENSION

For the application of an isotropic material law in the mesoscale model it is presumed that the finite element resolution of the heterogeneous solid on the mesoscale is very high. Then certain crack patterns can be reproduced by certain assemblies of isotropic subregions. Therefore the overall behaviour of the considered specimen can become essentially anisotropic. On the macroscale, modeling of such a material behaviour would require a more complex material law. An introduction to numerous material models, as well as to the isotropic model [9] presented in the following, is provided by *Jirásek* [8]. The considered model only includes damage in tension and does not include plasticity. This means that for complete unloading of the specimen the displacements return to zero. For quasibrittle materials such as concrete this presumption is not quite realistic. However, it is a prototype to examine the proposed grid-based concept.

In the isotropic material model, damage is described by the scalar damage parameter ω. If the material is not damaged then $\omega = 0$. The damage parameter ω can continuously increase up to 1, and $\omega = 1$ means that material is completely damaged. The damage parameter ω couples the linear elastic stress $\boldsymbol{\sigma}^e$ and the effective stress $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = (1 - \omega)\, \boldsymbol{\sigma}^e \tag{3.1}$$

There are several different measures of equivalent strain [8]. As one main criterion the equivalent strain shall lead to a realistic response in uniaxial tension. Here, the following definition is

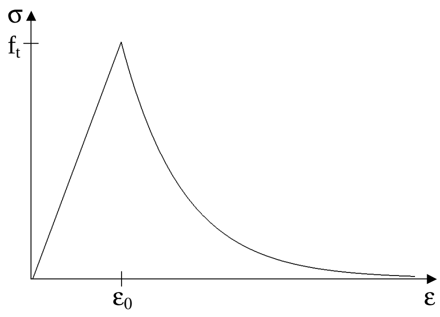

Fig. 3.1: Stress-strain diagram of damage model with exponential softening

selected.

$$\widetilde{\epsilon} = \frac{1}{E}\sqrt{\sum_{i=1}^{3} \langle S_i^e \rangle^2} \tag{3.2}$$

where E is the Young's modulus and $\boldsymbol{S}^e$ denotes the principal stresses of $\boldsymbol{\sigma}^e$. The brackets $\langle S_i^e \rangle$ mean the positive part of S_i^e or symbolically $\langle S_i^e \rangle = \max(0, S_i^e)$. Equation 3.2 refers to the three-dimensional case. For the plane stress problem, it can be reduced to

$$\widetilde{\epsilon} = \frac{1}{E}\sqrt{\langle S_1^e \rangle^2 + \langle S_2^e \rangle^2} \tag{3.3}$$

with the descriptive derivation of S_1 and S_2 from the Mohr circle as

$$S_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{3.4}$$

The maximum equivalent strain in the history of a material point is stored in κ. The damage parameter ω is defined by

$$\omega = \begin{cases} 0 & if \quad \kappa \leq \epsilon_0 \\ 1 - \dfrac{\epsilon_0}{\kappa} \exp\left(-\dfrac{\kappa - \epsilon_0}{\epsilon_f - \epsilon_0}\right) & if \quad \kappa > \epsilon_0 \end{cases} \tag{3.5}$$

where ϵ_0 is the limit elastic strain under uniaxial tension which is related to the tensile strength f_t by

$$\epsilon_0 = \frac{f_t}{E} \tag{3.6}$$

The parameter ϵ_f controls the ductility of the material in terms of the exponential softening branch (Fig. 3.1). In contrast, a linear softening branch tends to exhibit a defective snap-back and therefore is not proposed [8]. In Voigt notation the linear elastic material matrix $\boldsymbol{C}^e$ establishes the relationship between stress vector $\boldsymbol{\sigma}^e$ and strain vector $\boldsymbol{\epsilon}$.

$$\boldsymbol{\sigma}^e = \boldsymbol{C}^e \boldsymbol{\epsilon} \tag{3.7}$$

With the equality $\boldsymbol{C}(E, \nu) = E\boldsymbol{C}(1, \nu)$ where ν is the Poisson's ratio, Eq. 3.1 corresponds to $\boldsymbol{\sigma} = E\,(1 - \omega)\,C(1, \nu)\boldsymbol{\epsilon}$ which allows to define a degraded Young's modulus $\hat{E}$ as

$$\hat{E} = E\,(1 - \omega) \tag{3.8}$$

The equivalent strain and damage parameter are stored as an additional layer on the finite element grid (Section 2). A corresponding data field is also created for the degraded Young's modulus. Then the data field of E relates to the initial linear elastic stiffness and $\hat{E}$ to the secantial stiffness of the damaged material. It is noted that this is simply possible as the Poisson's ratio is not modified by this material model.

4 NONLOCAL FORMULATIONS

4.1 Motivation

A nonlocal formulation in this context means that a value of a field variable at any point will be influenced by the value of another, neighbouring point, and reverse. This is established by weighted averaging over source points of a predefined local domain around the considered material point, denoted as effect point. Such a nonlocal averaging can refer to various quantities. There are the following reasons from different perspectives to introduce nonlocal formulations in the considered numerical damage analysis:

a.1 By a local formulation of the damage model only, the damaged region could be infinitely small. Then the energy which is dissipated during the damage process would be equal to zero. The resulting ideally brittle behaviour of the material would be pathologic. From a mathematical point of view this defect results from a loss of ellipticity in the governing differential equations [8]. Therefore a localization limiter has to be introduced.

a.2 The physical experiment also indicates a nonlocal model. For example in the damage process of materials the development of crack bands or damage regions is observed. This effect is assigned to an internal length of the material.

a.3 A discrete numerical model will generally converge to the analytical solution and therefore the reasons for a nonlocal formulation of (a.1) also apply to finite element modeling. Without localization limiter, a defective sensitivity to the size of finite elements would be observed. Besides there are comfortable side-effects of nonlocal formulations in finite element modeling. Nonlocal averaging can be applied to recover strains or stresses, which is useful especially for the proposed grid-discretization.

The items (a.1) to (a.3) refer to the defect of strain localization from a rather mathematical, physical and numerical point of view. There are several solutions to this problem. In regularization techniques the material law is adapted such that the fracture energy is invariant to the size of finite elements, e.g. [11]. Another important category of solutions refers to nonlocal formulations which is reviewed in [1].

4.2 Nonlocal formulation by a weighting function

The nonlocal formulation by a weighting function α is applied to an arbitrary function f. The weighted average is denoted by $\bar{f}$.

$$\bar{f}(\boldsymbol{x}) = \int_{\Omega} \alpha(\boldsymbol{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \tag{4.1}$$

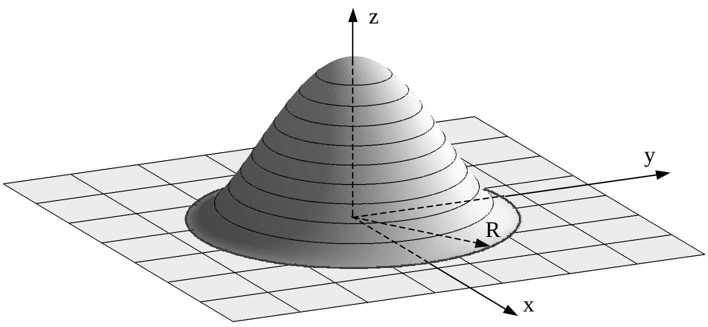

Fig. 4.1: Graph of bell-shaped function over two-dimensional domain with interaction radius R

The integral includes source points at coordinate $\boldsymbol{\xi}$ in the relevant domain Ω around the effect point $\boldsymbol{x}$. The effective weighting function $\alpha(\boldsymbol{x}-\boldsymbol{\xi})$ satisfies

$$\int_{\Omega} \alpha(\boldsymbol{x}-\boldsymbol{\xi}) = 1 \tag{4.2}$$

which is achieved by scaling of an initial weighting function $\alpha_0(\boldsymbol{x}-\boldsymbol{\xi})$ by

$$\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\alpha_0(\boldsymbol{x}-\boldsymbol{\xi})}{\displaystyle\int_{\Omega} \alpha_0(\boldsymbol{x}-\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}} \tag{4.3}$$

Equation 4.2 establishes that the nonlocal average $\bar{f}(\boldsymbol{x})$ and the function $f(\boldsymbol{x})$ are identical if the function $f(\boldsymbol{x})$ is constant. A favoured weighting function in the context of nonlocal damages models is the bell-shaped function. A two-dimensional representation of the bell-shaped function is shown in Fig. 4.1. With the general definition that r is the positive distance between the effect point and the source point $r = \|\boldsymbol{x}-\boldsymbol{\xi}\|$, it follows that the subsequent formulation of the bell-shaped function is valid for one, two or three dimensions.

$$\alpha_0(r) = \begin{cases} \left(1-\dfrac{r^2}{R^2}\right)^2 & \text{if } 0 \leq r \leq R \\ 0 & \text{if } R \leq r \end{cases} \tag{4.4}$$

In order to compare various weighting functions an internal length ℓ is introduced which is defined by the radius of inertia of the weighting function.

$$\ell = \sqrt{\frac{\displaystyle\int_0^{\infty} r^2\alpha_0(r)\mathrm{d}r}{\displaystyle\int_0^{\infty} \alpha_0(r)\mathrm{d}r}} \tag{4.5}$$

Another common weighting function is the Gauss distribution function which already includes the internal length ℓ by definition.

$$\alpha_0(r) = \exp\left(-\frac{r^2}{2\ell^2}\right) \tag{4.6}$$

A further relevant measure is the characteristic length ℓ_c which is defined by

$$\ell_c = \frac{\displaystyle\int_{-\infty}^{\infty} \alpha_0(r)\mathrm{d}r}{\alpha_0(r=0)} = \frac{1}{\alpha(r=0)} \tag{4.7}$$

For the damage analysis of heterogeneous materials the internal length ℓ and the characteristic length ℓ_c have a certain interpretation in regard to the (heterogeneous) character of the material. The ratio of ℓ_c/ℓ depends on the type of weighting function. It will be fixed, but generally different from 1 [8]. As final remark to this section of defining nonlocal weighting in general, it is pointed out that B-splines, beyond the application as finite element shape functions [6], in assignment to a polar coordinate system would also define adequate weighting functions.

4.3 Nonlocal formulation for corrective postprocessing of linear elastic, grid-based finite element solution

Independent of a damage material law, nonlocal formulations can be applied to improve the strains and stresses of a linear elastic finite element solution. Within this context the meaning of nonlocal formulations is extended to averaging techniques in general. It is highlighted that the underlying finite element analysis is still based on local operators. Averaging is only applied to reduce local errors in its solution. There are the following characteristics of emerging errors in finite elements:

b.1 In displacement-based finite elements the classical discretization error is related to limited approximation quality of shape functions due to limited order or limited resolution of finite elements.

b.2 The orientation of the finite element influences the error distribution. A systematic element orientation supports a systematic error with respect to a certain direction, especially in terms of strains and stresses.

b.3 It is an inherent characteristic of grid-based models that smooth surfaces of material phases are transformed into an angular discretization which will always lead to severe defects in the stress and strain solution independent of element order or resolution.

As in the present paper a nonlocal formulation of damage modelling is applied anyway, nonlocal averaging of the linear elastic solution does not increase the fuzziness, but may significantly improve the defects according to (b.1) to (b.3). Especially, the error according to (b.3), the grid-based discretization, can effectively be corrected. Therefore the grid-based catches up in comparison to more accurate aligned meshing[2]. There are various nonlocal formulations to improve the strain or stress solution of finite elements:

c.1 The use of a weighting function,

c.2 averaging of element values at nodes and

[2]However, while the stress solution can be improved, the effective energy error persists. With respect to the energy error higher-order elements would provide a relevant enhancement (as e.g. in [6]), as simple bilinear elements are known to be subject of severe locking effects. Nevertheless the correction of an error from grid-discretization can principally also be applied to higher-order elements. Furthermore it needs to be considered that by definition the finite element resolution is very high to represent the heterogeneous geometry.

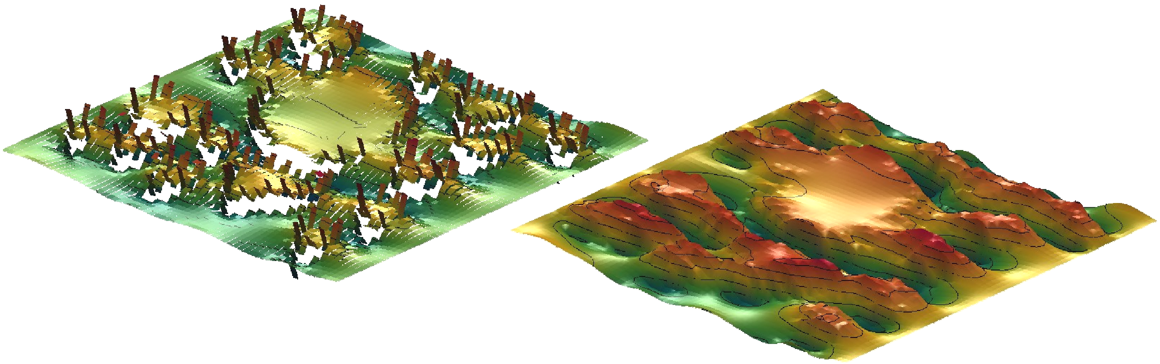

Fig. 4.2: Stress solution σ_{xx} before (on the left) and after nodal interpolation of element stress at nodes (on the right) for uniaxial load of material sample (Fig. 2.1) in x-direction.

c.3 special techniques such as patch recovery based on selected superconvergent points.

Nonlocal averaging according to (c.1) will automatically be included in the subsequent nonlocal damage formulation. However, it is reasonable to reduce the error mainly induced by grid-based discretization in advance. Figure 4.2 shows the effect on nodal averaging of the stress solution σ_{xx}. Similar good results are achieved for σ_{yy} and σ_{xy} (without illustration). The corrected stress solution is continuous and also the gradients as well as isolines of stresses are recovered. Figure 4.3 shows the isolines of the stress solution according to Fig. 4.2 (left); and Fig. 4.4 that of Fig. 4.2 (right). In Figure 4.3 it is obvious that almost all isolines are aligned horizontally. Although the nonlocal formulation of the damage will smooth the defect, it is assumed that an improved stress according to Fig. 4.4 will further reduce the mesh directional bias (the current implementation does not yet include this correction.).

As a further and rather unusual treatment of the problem an averaged material representation is introduced. Therewith the original problem is replaced by a substitute problem similar as described in [6]. Anyway, the substitute problem is only introduced to improve the stress solution in grid-based modelling and shall principally correspond to the same problem. It is noted that an averaged material representation is achieved by nonlocal averaging of Young's modulus, but, here, only one Young's modulus is assigned to each element. The effect of this prior averaging in the material model is seen in Fig. 4.5. From there it is concluded that good recovery of

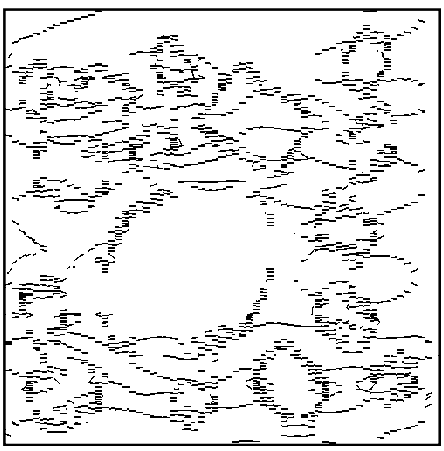

Fig. 4.3: Isolines of stress σ_{xx} in Fig. 4.2, left.

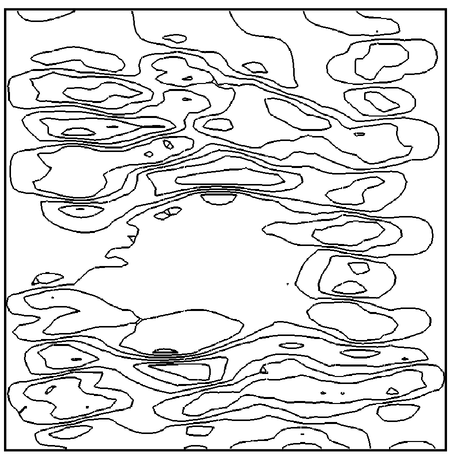

Fig. 4.4: Isolines of stress σ_{xx} in Fig. 4.2, right.

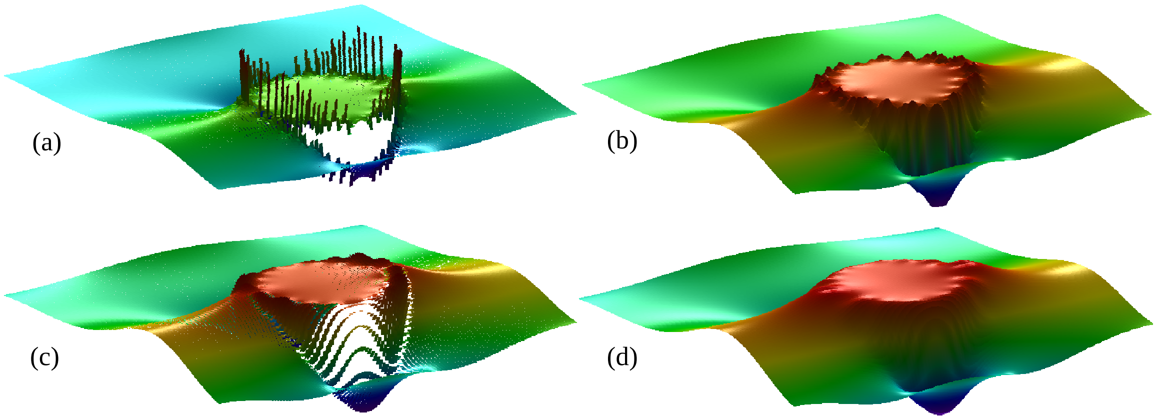

Fig. 4.5: Normal stress in direction of uniaxial tension of circular inclusion on uniform grid of bilinear finite elements: (a) direct element solution, (b) stress interpolated at nodes, (c) stress of model with nonlocal Young's modulus, (d) a combination of (b) and (c).

stress values is achieved by nodal averaging. An additional slight smoothing of Young's modulus might further improve the result, but, clearly, this option needs to be treated carefully. It is rather included as an interesting option which can, in this or in other context, be useful for some purpose.

4.4 Nonlocal formulation of damage law

For the nonlocal formulation of the considered damage law, the local equivalent strain $\tilde{\epsilon}$ of Eq. 3.3 is replaced by its nonlocal counterpart. In the finite element model the integral of Eq. 4.1 is replaced by a sum on discrete points according to

$$\bar{f}(\boldsymbol{x}) = \frac{\displaystyle\sum_{i \in A(\boldsymbol{x})} \alpha_0(\boldsymbol{x} - \boldsymbol{\xi}_i) f(\boldsymbol{\xi}_i)}{\displaystyle\sum_{i \in A(\boldsymbol{x})} \alpha_0(\boldsymbol{x} - \boldsymbol{\xi}_i)} \tag{4.8}$$

The sequence $A(\boldsymbol{x})$ only includes source points $\boldsymbol{\xi}_i$ which are within the same individual phase (Fig. 2.1 (c)) as effect point $\boldsymbol{x}$ to avoid averaging over different materials. It follows that the denominator in Eq. 4.8 can be different for different $\boldsymbol{x}$. In the present implementation the coordinates of source points refer to the center of elements[3]. It is reasonable to restrict element size h in relation to the interaction radius R of the weighting function by $h \leq \frac{R}{3}$[8].

On the defined uniform, orthogonal grid the weighting function can be applied as a predefined discrete stencil, which can arbitrarily be trimmed along material or model boundaries. Therefore nonlocal formulations on uniform grids are quite effective. Furthermore as the local equivalent strain only includes the positive parts of stresses, it appears more accurate to apply nonlocal averaging already to the stresses in Eq. 3.4. From this it follows that there are several possible variations of the method which need to be considered for achieving an optimal result.

[3]The stresses at the center of bilinear rectangular finite elements are denoted superconvergent. However, in combination with nodal averaging of Section 4.3, a more accurate numerical integral based on nodal values would be useful to reduce the mesh dependency as best as possible.

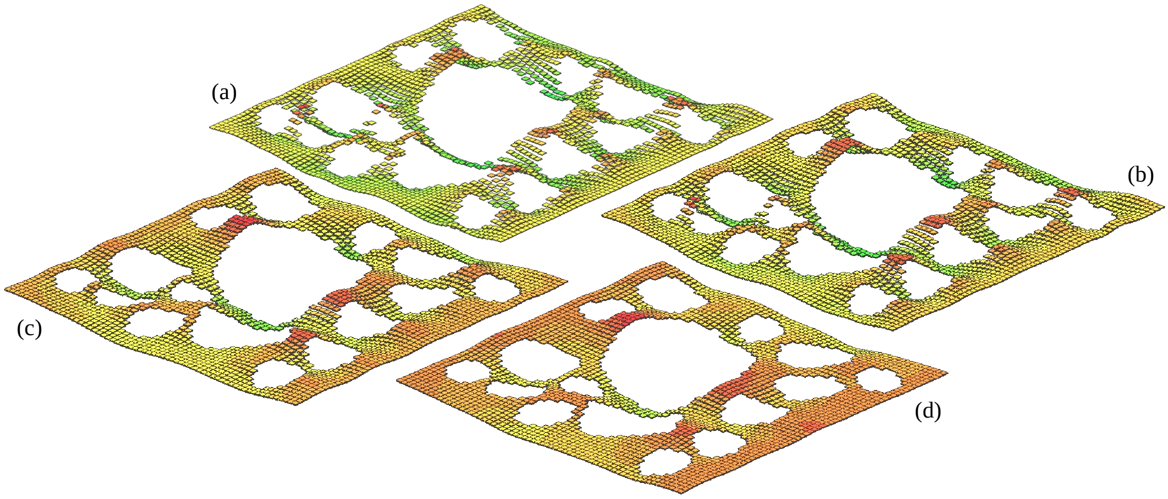

Fig. 4.6: Nonlocal equivalent strain in center of elements according to bell-shaped function with (a) R=1, (b) R=5, (c) R=10 and (d) R=15 of specimen in Fig. 2.1 with side length 100.

4.5 Nonlocal formulation in adaption to heterogeneous material

Besides stabilizing the numerical mechanical analysis, the nonlocal formulation refers to the physical phenomenon of damage in material. Specific crack band width or size of damage region occurs in different materials. It depends on a specific internal length of each material. For heterogeneous materials it essentially results from the microstructure. Then, the type and size of weighting function not only determines the smoothing radius of post-values as shown in Fig. 4.6, but will also define the size, as well as the curvature, of the developing damage region. In this regard, various weighting functions, such as the Gauss distribution function and the bell-shaped function, show perceivable differences [8].

To track the physical phenomenon, concrete material is selected. It is a comprehensible explanation that material points in the closest neighborhood around an inclusion in a generally weaker matrix will not be damaged independently of each other. In this perception the aggregate will ”distribute” the damage. From a combined experimental and numerical approach, it is adopted in [3] that the material characteristic length is roughly equal to the maximum aggregate size. In a numerical study in [2] the interaction radius is set to the triple of the maximum aggregate size which also confirms the principle idea. With respect to damage modeling on the mesoscale, the relevant size may be related to the largest aggregate size which is not explicitly modeled, as included aggregates will interact intuitively.

4.6 Nonlocal formulation for grid-based discretization of material interphases

The present approach discusses the possibility of a purely grid-based approach for ideal data structures, optimal embedding into the multigrid method and extension to three-dimensional modeling. In this context a possibly novel idea for modelling material interphases on a grid shall only be mentioned.

Besides the stress values also the stress directions can be recovered. By a proper definition of a multiphase material also the orientation of a (one-pixel thick) interphase can be approximated by a nonlocal formulation, as indicated in Fig. 4.7. With both strain and interphase directions being approximated a special interface solution can be applied, e.g. isotropic softening of interface elements is defined only for positive normal stress perpendicular to the orientation of the interface. A corresponding method requires some kind of regularization with respect to element

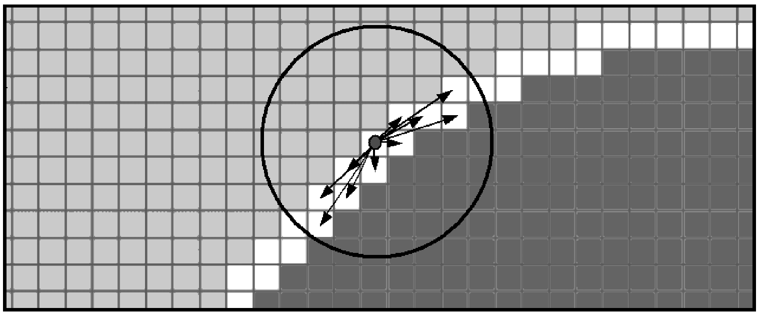

Fig. 4.7: Interpolation of directions to approximate the interphase orientation based on plain grid discretization

size, another weighting function for directions, as well as several additional considerations. It is highlighted that this concept presumes a very high resolution of the material, which is just the intention of grid-based modelling.

5 DAMAGE-CONTROLLED SEQUENTIALLY LINEAR ANALYSIS

Damage processes describe a nonlinear behaviour. There are various iteration procedures to solve the corresponding nonlinear equations. Load-controlled methods will fail to converge after an ultimate load is reached and a global softening behaviour can not be simulated. A displacement-controlled analysis enables to step beyond an ultimate loading. However, a snap-back can not be detected by displacement-control. Snap-back behaviour describes the phenomenon that for equilibrium on the nonlinear path, both, the load as well as the displacement will be reduced. Such a behaviour can refer to an actual brittle behaviour of a material or structure. But it will also occur due to an unrealistic linear softening branch in the material model or unregularized strain localization in the numerical model [8]. The arc-length method is an advanced iteration procedure which is capable to cover post-peak behaviour as well as snap-back. For example, recently the arc-length method has been demonstrated for simulating of such damage processes in concrete by *Most* [10].

The present article follows a sequentially linear approach similar to *Rots & Invernizzi* [11]. This method mitigates the problem to achieve a stable numerical iteration procedure. In fact, each iteration step is performed by linear analysis utilizing the secantial stiffness matrix. The sequentially linear approach covers post-peak behaviour, snap-back and is any case stable, at least from a numerical point of view. The algorithm is outlined by the following steps:

1. Perform a linear elastic analysis with unit load.
2. Determine a critical load factor for a small increase of damage, e.g. such that exactly one element of the model will pass the peak point in the stress-strain diagram.
3. Scale displacement solution according to critical load factor and record actual status of load and displacement.
4. Update damage variables according to this displacement solution of the last load factor.
5. Build a new secantial stiffness matrix. Here, instead, for grid-based models and the scalar damage law, only a modified field of Young's modulus is stored. Repeat procedure.

The linear elastic analysis in each new iteration is based on the degraded field of Young's modulus. However, the initial field of Young's modulus is still required to update the damage variables. This iteration scheme is declared as damage-controlled. In each step a small additional, controlled damage will occur within the actual critical region or regions. If the additional damage is most relevant only in one element which passes the linear elastic limit, then it is assumed that the damage will follow the correct path. However, there are other possible measures for the amount of damage or increase of damage. Anyway, if the additional damage steps are small enough, the same path will be followed towards the actual critical region(s). In combination with efficient iterative solvers such as the multigrid method [5], always the solution of the last step can be applied as the start vector of the actual step. Therefore a decrease of the damage step size results in a decrease of iteration time per step. Most probably the effective computation time will still increase if the step size is reduced, but for further investigations it appears quite interesting to examine this trade-off between more damage steps, but less computational effort to reach equilibrium for each step.

From the documented algorithm it follows that in each iteration step the actual load limit is a little bit overestimated, as it is already used to initiate a small damage increment for the next iteration step. However, it also simply possible to achieve exact equilibrium. Then, for the actual damage situation the linear secantial solution is scaled by a load factor just that in no element additional damage will occur. Then this describes the maximum load for which the current damage state is still exactly valid.

Rots and Invernizzi [11] describe another criterion. The linear secantial solution is scaled such that the stress in the most critical element corresponds to the ultimate stress. In the present approach, it needs to be decided to consider the local or an equivalent nonlocal stress. However, only if the damage does not increase in any element, as described in the foregoing paragraph, then the equilibrium is exact. Therefore the stress criterion appears less accurate from the present point of view. However, in the following examples such a criterion of equivalent nonlocal stress has been applied.

Another tested criterion to induce additional damage is that the maximum strain which is detected in any of the elements will be increased by a predefined factor just above 1. Or additional damage can be initiated based on a global measure of damage. Nevertheless, any of these measures can be reduced to an over-estimation of the actual valid load factor on the actual secantial stiffness to continue the damage process. If the steps of any measure will be small enough, then there will be no (relevant) difference in the damage path.

In this regard the most relevant aspects of the damage-controlled sequentially linear analysis refer to the step size of additional damage in each step. However, we assume that the simulation of complex damage processes on the material level always requires many steps to detect the actual damage path which is sensitive to multiple bifurcation, e.g. a rough curvature, such as a crack around numerous inclusions can not accurately be approximated in a few steps by any method. In the proposed approach the computational effort of each step is minimal in comparison to other iteration procedures. Nevertheless, in further investigations the effective accuracy needs to be analyzed carefully. In the ideal case, an objective measure can be applied to control step size of damage and effective accuracy of each step.

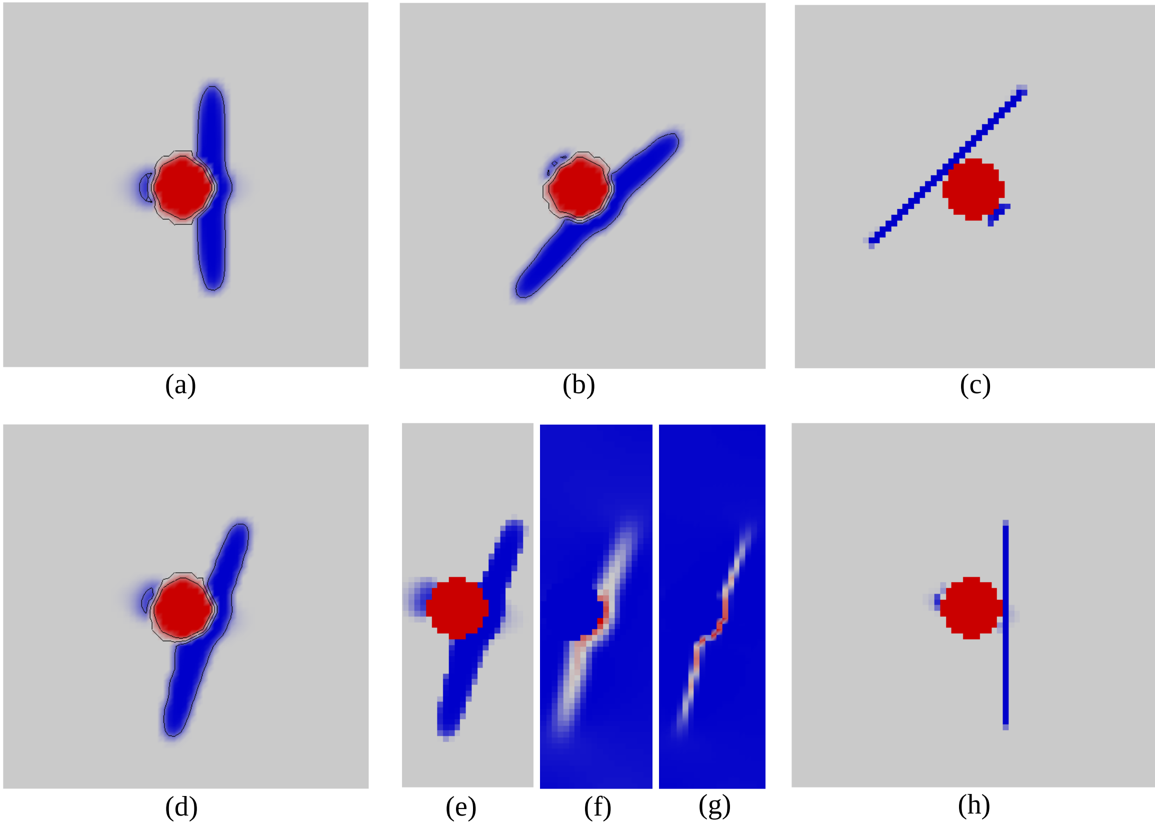

Fig. 6.1: Plane stress problem of circular inclusion (16mm) in quadratic plate (100mm, 64x64 finite elements) under uniaxial tension modeled by 64x64 finite elements. Ratio of Young's modulus of inclusion to matrix is 2:1. Nonlocal radius of bell-shaped function is 5mm. All images except (f) and (g) show the degraded Young's modulus according to damage. (a) horizontal tension, (b) diagonal tension, (c) as (b) but without nonlocal weighting, (d) direction of applied tension between (a) and (b) with slight deviation of damage orientation, (e) same as (d) but graphical output is not interpolated, (f) nonlocal equivalent strains according to (e), (g) equivalent strains, (h) same loading as (d) but severe deviation from angle without nonlocal formulation.

6 NUMERICAL EXAMPLE

This article includes one example to analyze mesh directional bias of uniaxial tension tests on a uniform orthogonal grid. The results are illustrated in Fig. 6.1. From Fig. 6.1 (d) it follows that to a large extent mesh dependency has been reduced by the nonlocal formulation in comparison to the local formulation of Fig. 6.1 (h). However, in Fig. 6.1 (d) the damage region does not exactly develop perpendicular to the applied direction of uniaxial tension. This can result from the large expansion of the damage region while the specimen is asymmetric to the corresponding direction of applied uniaxial tension. Furthermore an improvement might be achieved by nodal averaging of stresses as described in Section 4.3.

Additionally the load-displacement relationship for maximum displacement in the example of Fig. 6.1 (a) has been recorded in Fig. 6.2. The test has been performed for three different interaction radii R of the weighting function. With increasing interaction radius R the area below the corresponding softening branch increases which can be ascribed to an increased fracture energy. Furthermore also the ductility increases which is interpreted from the smaller gradient of the softening branch after the peak load. This example illustrates some preliminary results of the proposed method. Further examples will be presented at the conference.

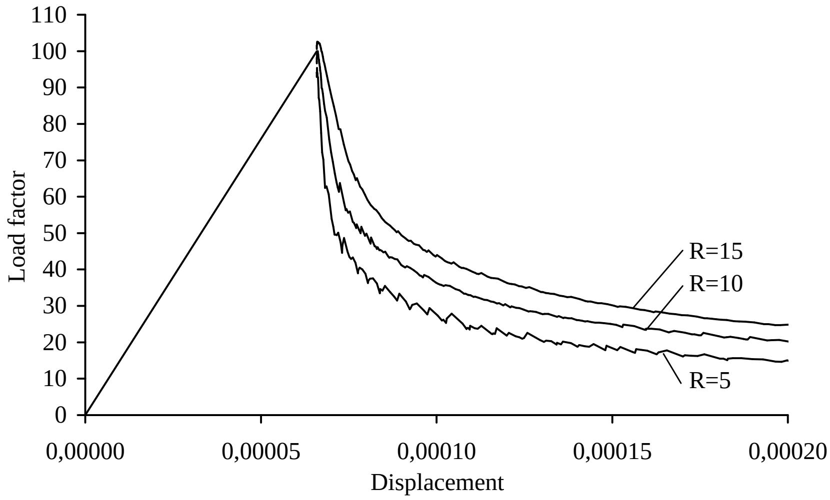

Fig. 6.2: Load-displacement diagram for example of Fig. 6.1 (a). The displacement is recorded from the center of the right side of the specimen and which, according to load and support, will correspond to the maximum displacement.

7 CONCLUSIONS

The article discusses nonlocal damage modelling by an isotropic material law with exponential softening for tension. The application bases on the assumption that the resolution of the numerical model is very high such that crack patterns of a region can be reproduced by an assembly of much smaller isotropic subregions. The theoretical basis of such damage simulations is prepared and directly linked to the approach of grid-based modelling. It is confirmed that severe defects of stress from the grid-based, linear elastic finite element solution can significantly be improved by nonlocal postprocessing. By these corrections one disadvantage of grid-based modelling is significantly reduced. On the other hand, the uniform grid supports an effective implementation of nonlocal averaging by predefined discrete data of the weighting function. A damage-controlled procedure of linear steps is performed, which means that from one step to another only a certain increment of damage occurs. Several parameters need to be considered for validation of this promising method. Considerable efficiency has been demonstrated for the grid-based linear elastic analysis of heterogeneous material samples in [5]. In this idea, the article presents the first successful steps to extend the effective grid-based principle to nonlinear damage simulation of materials and prepares for a further development of the grid-based approach, in addition to [6, 7].

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